

**EE4600 - Control Systems Design**  
Assignment 1

Sven Richter

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# 1 Introduction

In figure 1 a regulator control system is shown, in which it is desired to maintain the output  $c(t)$  at a value of zero in the presence of a disturbance  $f(t)$ . In this problem the disturbance is a unit step.

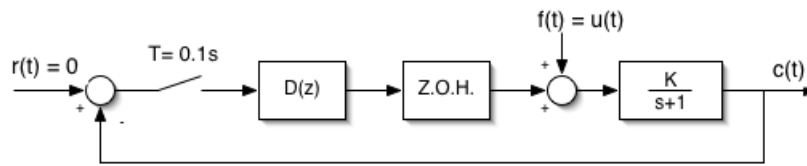


Figure 1: Regulator control system

1. With  $D(z) = 1$  (ie no compensation) and an open loop gain  $K=1$ , find the steady state value of  $c(t)$ . Use a simulink simulation to verify your results.
2. Devise a controller  $D(z)$  that will ensure that the system output  $c(t)$  will return to zero following the disturbance. The response should be second order with an overshoot of approximately 30%. Justify your choice of controller and verify its performance in simulink.
3. Determine the range of open loop gain  $K$  for which the system of part b will be stable.
4. Draw the root locus plot for the system of part b and using the plot, show how the system closed loop poles and hence its response will vary as the sampling period  $T$  is increased. Verify your answer using simulink.

## 2 Solution

### 2.1 Transfer function of the System

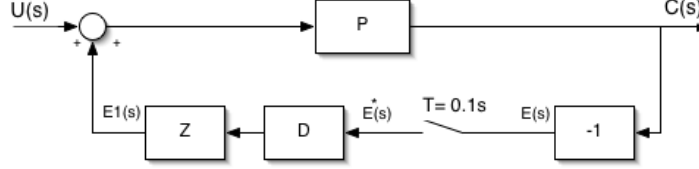


Figure 2: System

For the system shown in figure 2 the identifiers are defined as following:

**P** Transfer function of the plant  $\frac{K}{s+1}$

**Z** Transfer function of the zero order hole  $\frac{1-e^{-sT}}{s}$

**D** Transfer function of the controller

From the system diagram the following equations can be written off:

$$C(s) = P(s)(U(s) + E_1(s)) \quad (1)$$

$$E_1(s) = Z(s)D(s)E^*(s) \quad (2)$$

$$E^*(s) = -C(s) \quad (3)$$

Putting equation 2 and 1 into equation 3 we get the following equation.

$$E^*(s) = -P(s)U(s) - P(s)Z(s)D(s)E^*(s) \quad (4)$$

Equation 4 can now be star-transformed.

$$\begin{aligned} E^*(s) &= -\overline{P(s)U(s)}^* - \overline{P(s)Z(s)D(s)}^* E^*(s) \\ E^*(s) + \overline{P(s)Z(s)D(s)}^* E^*(s) &= -\overline{P(s)U(s)}^* \\ E^*(s) \left(1 + \overline{P(s)Z(s)D(s)}^*\right) &= -\overline{P(s)U(s)}^* \\ E^*(s) &= -\frac{\overline{P(s)U(s)}^*}{1 + \overline{P(s)Z(s)D(s)}^*} \end{aligned} \quad (5)$$

If we take equation 2 and put it into equation 1 we get an expression for  $C(s)$

$$C(s) = P(s)U(s) + P(s)Z(s)D(s)E^*(s) \quad (6)$$

Equation 6 will also be star transformed.

$$C^*(s) = \overline{P(s)U(s)}^* + \overline{P(s)Z(s)D(s)}^* E^*(s) \quad (7)$$

Now equation 5 is put into 7

$$C^*(s) = \frac{\overline{P(s)U(s)^*} - \overline{P(s)Z(s)D(s)^*} \frac{\overline{P(s)U(s)^*}}{1 + \overline{P(s)Z(s)D(s)^*}}}{1 + \overline{P(s)Z(s)D(s)^*}}$$

$$C^*(s) = \frac{\overline{P(s)U(s)^*} \left(1 + \overline{P(s)Z(s)D(s)^*}\right)}{1 + \overline{P(s)Z(s)D(s)^*}} - \frac{\overline{P(s)Z(s)D(s)^*} \overline{P(s)U(s)^*}}{1 + \overline{P(s)Z(s)D(s)^*}}$$

$$C^*(s) = \frac{\overline{P(s)U(s)^*} + \overline{P(s)U(s)^*} \overline{P(s)Z(s)D(s)^*} - \overline{P(s)Z(s)D(s)^*} \overline{P(s)U(s)^*}}{1 + \overline{P(s)Z(s)D(s)^*}}$$

$$C^*(s) = \frac{\overline{P(s)U(s)^*}}{1 + \overline{P(s)Z(s)D(s)^*}} \tag{8}$$

$$\tag{9}$$

With equation 8 it is possible to formulate the Z-transform of  $C$  as

$$C(z) = \frac{\overline{P * U}(z)}{1 + \overline{P * Z * D}(z)} \tag{10}$$

## 2.2 Steady state value of $c(t)$

### 2.2.1 Calculation

The Steady state value of the system should be calculated for  $K = D = 1$  and a step input. Therefore the Parts of the system are defined as:

$$U(s) = \frac{1}{s} \quad (11)$$

$$P(s) = \frac{1}{s+1} \quad (12)$$

$$D(s) = 1 \quad (13)$$

$$Z(s) = \frac{1 - e^{-sT}}{s} \quad (14)$$

With statements 11 to 14 on the current page we get the following expressions.

$$P(s)U(s) = \frac{1}{s(s+1)} \quad (15)$$

$$P(s)Z(s)D(s) = \frac{1 - e^{-sT}}{s(s+1)} \quad (16)$$

Equation 15 can also be written as

$$P(s)U(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \quad (17)$$

The z-transformation of equation 17 can now be done with the tables.

$$\begin{aligned} \overline{P * U}(z) &= \frac{z}{z-1} - \frac{z}{z - e^{-T}} \\ \overline{P * U}(z) &= z \frac{z - e^{-T} - z + 1}{(z-1)(z - e^{-T})} \\ \overline{P * U}(z) &= \frac{z(1 - e^{-T})}{(z-1)(z - e^{-T})} \end{aligned} \quad (18)$$

Equation 16 can also be written as:

$$P(s)Z(s)D(s) = \frac{1}{s(s+1)} (1 - e^{-sT}) \quad (19)$$

The Z-transformation of  $\frac{1}{s(s+1)}$  is already known from equation 18. So only the Z-transformation of  $1 - e^{-sT}$  must be found. In the s-domain  $e^{-sT}$  is just a delay of T. In the z-domain a delay of T (sampling time) is expressed as  $z^{-1}$ . Therefore the the z-transformation of  $1 - e^{-sT}$  is  $1 - z^{-1}$ .

$$1 - z^{-1} = \frac{z-1}{z} \quad (20)$$

With equation 18 and 20 the z-transformation of equation 16 is derived as:

$$\begin{aligned}
\overline{P * Z * D}(z) &= \frac{z(1 - e^{-T})}{(z - 1)(z - e^{-T})} \frac{z - 1}{z} \\
\overline{P * Z * D}(z) &= \frac{1 - e^{-T}}{z - e^{-T}}
\end{aligned} \tag{21}$$

With equation 21 and 18 it is possible to formulate the z-transformed output function of the system.

$$\begin{aligned}
C(z) &= \frac{1}{1 + \frac{1 - e^{-T}}{z - e^{-T}}} \frac{z(1 - e^{-T})}{(z - 1)(z - e^{-T})} \\
C(z) &= \frac{(z - e^{-T})}{(z - e^{-T}) + (1 - e^{-T})} \frac{z(1 - e^{-T})}{(z - 1)(z - e^{-T})} \\
C(z) &= \frac{z(1 - e^{-T})}{z - e^{-T} + 1 - e^{-T}} \frac{1}{(z - 1)} \\
C(z) &= \frac{z(1 - e^{-T})}{z - 2e^{-T} + 1} \frac{1}{(z - 1)}
\end{aligned} \tag{22}$$

To derive the steady state value of  $c(t)$  the final value theorem can be used.

$$\begin{aligned}
c(\infty) &= \lim_{t \rightarrow \infty} c(t) = \lim_{z \rightarrow 1} (z - 1)C(z) \\
&= \lim_{z \rightarrow 1} (z - 1) \frac{z(1 - e^{-T})}{z - 2e^{-T} + 1} \frac{1}{(z - 1)} \\
&= \lim_{z \rightarrow 1} \frac{z(1 - e^{-T})}{z - 2e^{-T} + 1} \\
&= \frac{1 - e^{-T}}{2 - 2e^{-T}} \\
&= \frac{1}{2} \frac{1 - e^{-T}}{1 - e^{-T}} \\
&= \frac{1}{2}
\end{aligned} \tag{23}$$

As it can be seen in equation 23 the steady state value was derived to 0.5.

### 2.2.2 Verification with simulink

To check the calculated result the model was build in simulink (figure 3).

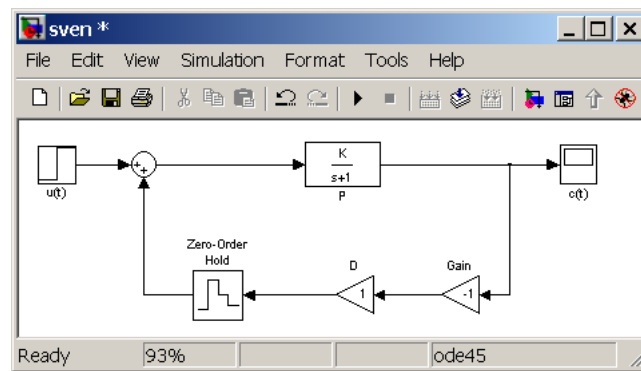


Figure 3: Simulink model

As it can be seen in figure 4 the calculated and experimental achieved results are equal.

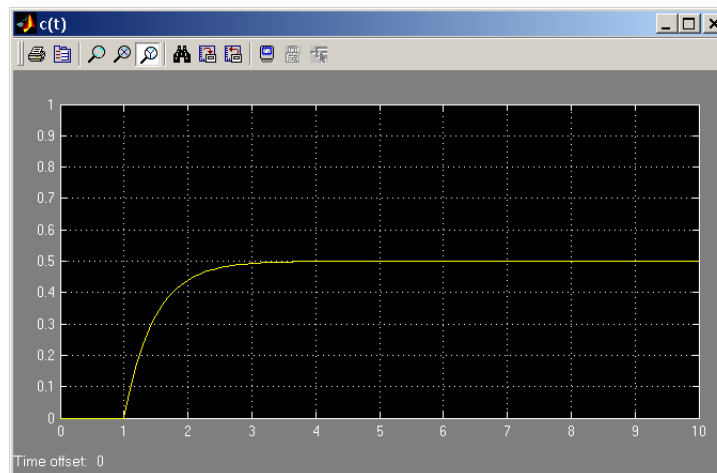


Figure 4: Step response



## 2.3 Controller design

To satisfy the demanded results, a steady state value of zero to an step input of one, D must contain an integrator. To achieve the overshoot of 30 % an additional P controller is added. Therefore we use an PI controller. So  $D(z)$  will be:

$$D(z) = \frac{zC}{z-1} \quad (24)$$

The output function then becomes to:

$$\begin{aligned} C(z) &= \frac{(z - e^{-T})(z - 1)}{(z - e^{-T})(z - 1) + CKz(1 - e^{-T})} \frac{zK(1 - e^{-T})}{(z - 1)(z - e^{-T})} \\ &= \frac{zK(1 - e^{-T})}{(z - e^{-T})(z - 1) + KCz(1 - e^{-T})} \\ &= \frac{zK(1 - e^{-T})}{(z^2 - ze^{-T} - z + e^{-T}) + z(CK - CKe^{-T})} \\ &= \frac{zK(1 - e^{-T})}{z^2 + z(-1 + CK - e^{-T}(CK + 1)) + e^{-T}} \end{aligned} \quad (25)$$

The value of C which satisfies the specifications was determined with the simulink model shown in figure 5.

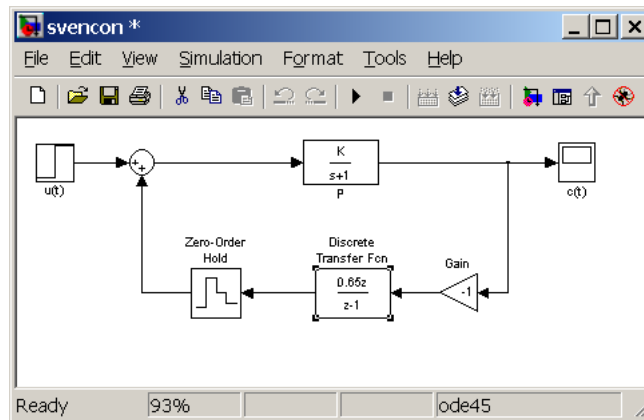


Figure 5: Simulink model with controller

This controller has an overshoot of approximately 30% if  $C=0.65$ . This can be seen in figure 6.

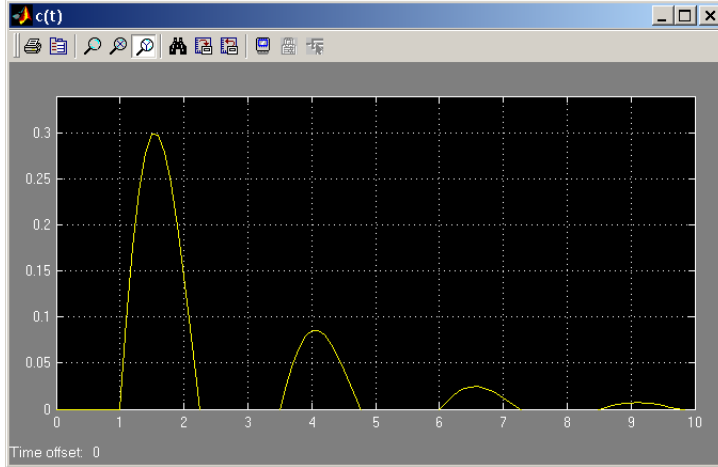


Figure 6: Step response with controller

## 2.4 Stability range of K

$$C(z) = \frac{zK(1 - e^{-T})}{z^2 + z(-1 + CK - e^{-T}(CK + 1)) + e^{-T}} \quad (26)$$

The system function (eq. 26) has pole at the following positions

$$\begin{aligned} 0 &= z^2 + z(-1 + CK - e^{-T}(CK + 1)) + e^{-T} \\ z_{1/2} &= -\frac{1}{2}(-1 + CK - e^{-T}(CK + 1)) \pm \sqrt{(-1 + CK - e^{-T}(CK + 1))^2 - e^{-2T}} \end{aligned}$$

As shown in the root locus plot in figure 7 on the following page is the system stable if the poles real part is greater than minus one and less than one. Because the system is stable if the poles are in the unit circle.

$$-1 \leq -\frac{1}{2}(-1 + CK - e^{-T}(CK + 1)) \leq 1$$

First we derive which value of K can be the maximum.

$$\begin{aligned} -1 &= -\frac{1}{2}(-1 + CK_{max} - e^{-T}(CK_{max} + 1)) \\ 2 &= -1 + CK_{max} - e^{-T}CK_{max} - e^{-T} \\ 3 + e^{-T} &= CK_{max} - e^{-T}CK_{max} \\ 3 + e^{-T} &= CK_{max}(1 - e^{-T}) \\ K_{max} &= \frac{3 + e^{-T}}{C(1 - e^{-T})} = 63.128 \end{aligned} \quad (27)$$

Now the lower value of K is calculated.

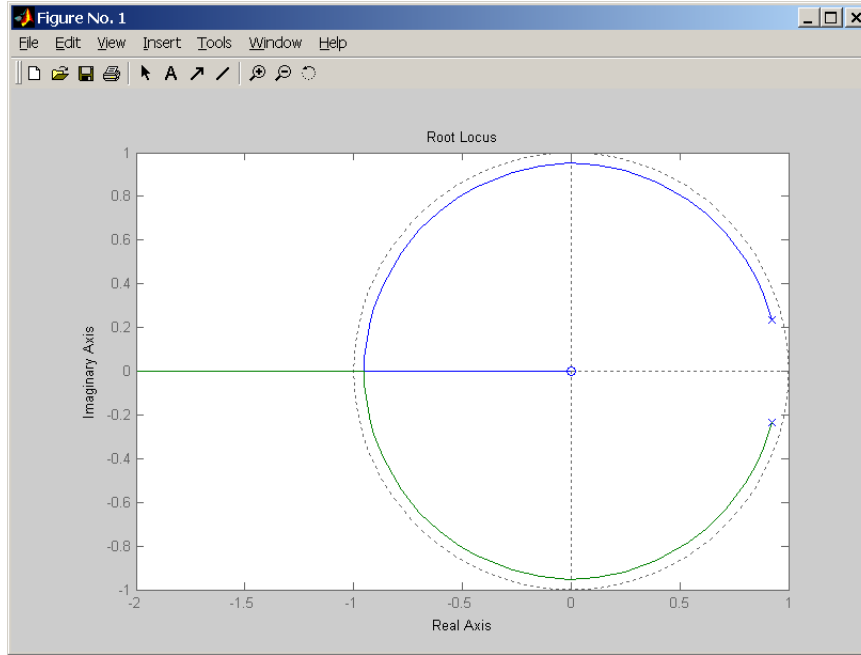


Figure 7: Root locus plot

$$\begin{aligned}
 1 &= -\frac{1}{2}(-1 + CK_{min} - e^{-T}(CK_{min} + 1)) \\
 -2 &= -1 + CK_{min} - e^{-T}CK_{min} - e^{-T} \\
 -1 + e^{-T} &= CK_{min} - e^{-T}CK_{min} \\
 -1 + e^{-T} &= CK_{min}(1 - e^{-T}) \\
 K_{min} &= \frac{-1 + e^{-T}}{C(1 - e^{-T})} = -1.53 \quad (28)
 \end{aligned}$$

Because K is the loop gain it must have a value greater than zero. Therefore is the stability range of K derived as;

$$\underline{0 < K \leq 63.1}$$

## 2.5 Variation of the sampling period

### 2.5.1 Root locus

If the sampling time increases the root locus plot becomes smaller. This is shown in figure 8 and 9. This means the absolute value of the poles decreases. Therefore the overshoot and settling time increases. But also the system is stable in a smaller area of  $K$ .

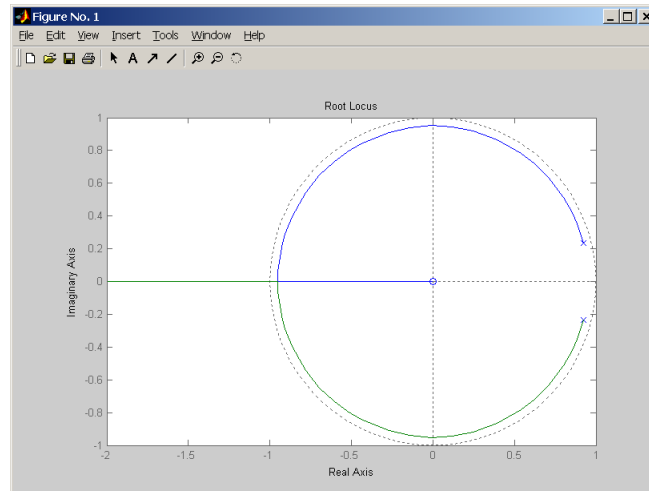


Figure 8: Root locus plot -  $T = 0.1s$

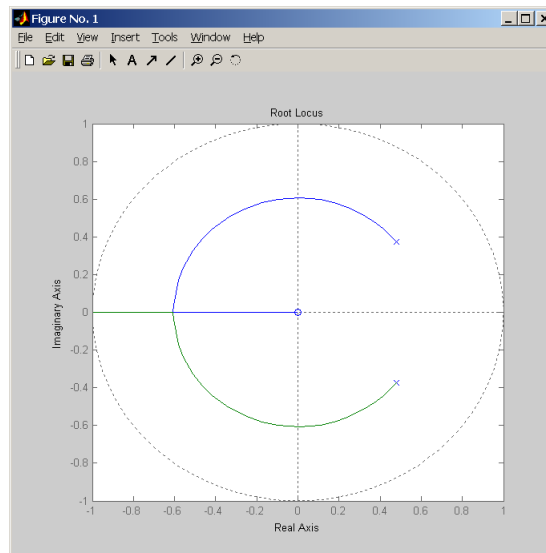


Figure 9: Root locus plot -  $T = 1.0s$

### 2.5.2 Simulink

The increase of settling time and overshoot can easily be observed in the step responses shown in figure 10 to 12 on the current page.

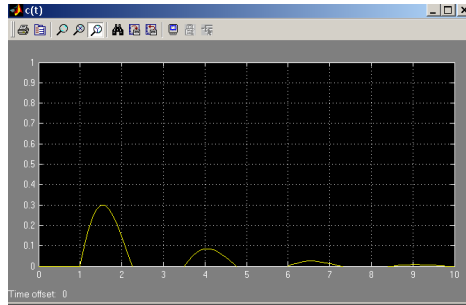


Figure 10: Step response -  $T = 0.1s$

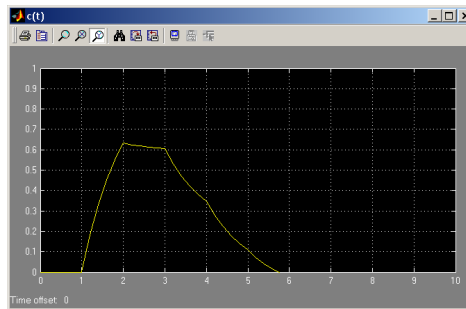


Figure 11: Step response -  $T = 1.0s$

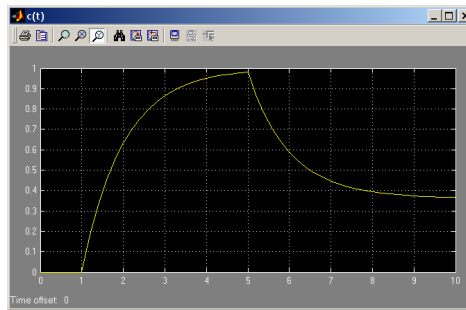


Figure 12: Step response -  $T = 5.0s$